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On the Range of a Fleet of Aircraft

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J. N. Franklin has determined the maximum range of the last member of a fleet of n aircraft with identical fuel capacities and gallons-per-mile fuel efficiencies, on the assumptions that in-air exchange of fuel is unrestricted and that aircraft can be abandoned from the fleet at any stage. The present paper extends the solution (a) to the case of aircraft with equal fuel efficiencies but different fuel capacities, and (b) to the case of aircraft with equal fuel capacities but different fuel efficiencies.

1. Introduction

In a previous paper,* J. N. Franklin considers the problem of determining the maximum range attainable by a fleet of n aircraft with an initial total supply of g gallons of fuel. It is assumed that the ith aircraft has a fuel capacity of g_i gallons and a fuel efficiency of r_i gallons per mile. The aircraft may share fuel in flight, and any of them may be abandoned at any stage.

Franklin gives an explicit solution for $n=2$, which illustrates a dynamic programing method available for larger n and also indicates the complexity of the solution in the general case. He also gives a complete discussion (including an asymptotic formula for g as a function of the maximum range) of the *symmetric case* in which

$$g_i=G, \quad r_i=R \qquad \text{for } i=1, 2, \ldots, n.$$

In the present paper, Franklin's solution for the symmetric case is extended (a) to the situation in which the aircraft of the fleet all have the same fuel efficiency, though their fuel capacities may differ, and (b) to the case of equal fuel capacities with (possibly) different fuel efficiencies.

2. Equal Fuel Efficiencies

In this section it will be assumed that

$$r_i=R \qquad \text{for } i=1, 2, \ldots, n \tag{1}$$

and that the aircraft are so numbered that

$$0<g_1\leq g_2 \ldots \leq g_{n-1}\leq g_n. \tag{2}$$

It is convenient to define

$$G_0=0; \quad G_k=\sum_{j=1}^{k} g_{n+1-j} \qquad \text{for } k=1, 2, \ldots, n; \quad G_{n+1}=\infty. \tag{3}$$

The maximum range will be denoted by $M_n(g)$, or more fully by

$$M(g; g_1, \ldots, g_n)=M_n(g). \tag{4}$$

It will be shown that, if k is the unique integer defined by

$$G_k<g\leq G_{k+1}, \tag{5}$$

then the maximum range is given by

$$M_n(g)=R^{-1}\left\{\frac{g}{k+1}+\frac{G_1}{1\cdot 2}+\frac{G_2}{2\cdot 3}+\cdots+\frac{G_k}{k(k+1)}\right\} \qquad \text{if } g\leq G_n, \tag{6}$$

$$M_n(g)=R^{-1}\left\{\frac{G_n}{n}+\frac{G_1}{1\cdot 2}+\frac{G_2}{2\cdot 3}+\cdots+\frac{G_{n-1}}{(n-1)n}\right\} \qquad \text{if } g>G_n. \tag{7}$$

In the optimal policy, if $g\leq G_n$, then the flight is begun with aircraft $n, n-1, \ldots, n-k$. If $g>G_n$, then $g-G_n$ gallons of fuel must be discarded and the flight is begun with all n aircraft. In any case, if the flight is begun with aircraft $n, n-1, \ldots, n-t$, then the last-listed of these aircraft is abandoned when only G_t gallons of fuel are left, the next-to-last is abandoned when only G_{t-1} gallons remain, etc.

This result will be proved by induction on n. If $n=1$, then formulas (6) and (7) become

$$M_1(g)=\min\,(g/R, g_1/R),$$

which is obviously true. Now assume the result holds for all fleets of $n-1$ aircraft, where $n>1$. Let $x\geq 0$ be the distance at which the first abandonment of an aircraft occurs. (Setting $x=0$ indicates that one or more aircraft are left at the base.) Then

$$M_n(g)=\max_{x\leq g/nR}\{x+\max_{1\leq j\leq n} M(g-nRx; g_1, \ldots, \hat{g}_j, \ldots, g_n)\} \tag{8}$$

*J. N. Franklin, The range of a fleet of aircraft, J. Soc. Ind. Appl. Math. **8** (1960), 541–548.

for $g \leq G_n$, where the caret denotes an omitted argument. The inductive hypothesis shows that

$$M(y; y_1, y_2, \ldots, y_{n-1})$$

is a non-decreasing function of its last $n-1$ arguments over the set

$$0 \leq y_1 \leq y_2 \leq \ldots \leq y_{n-1},$$

and so, using (2), (8) becomes

$$M_n(g) = \max_{x \leq g/nR} \{x + M(g - nRx; g_2, \ldots, g_n)\} \quad (9)$$

for $g \leq G_n$. The inductive hypothesis shows that the polygonal function

$$P(x) = x + M(g - nRx; g_2, \ldots, g_n) \quad (10)$$

of x, at all points in $0 < g - nRx < G_{n-1}$ for which it has a derivative, has a *negative* derivative, and so $P(x)$ is decreasing for $0 < g - nRx < G_{n-1}$.

If $g \leq G_{n-1}$ then the maximum of $P(x)$ required in (9) is therefore obtained by setting $x = 0$, yielding

$$M_n(g) = M(g; g_2, \ldots, g_n). \quad (11)$$

But by the inductive hypothesis, the right side of (11) is given by the right side of (6) for $g \leq G_{n-1}$. Thus (6) is proved for $g \leq G_{n-1}$.

Next suppose that $G_{n-1} < g \leq G_n$. The polygonal function $P(x)$ is decreasing for $0 \leq g - nRx \leq G_{n-1}$, as noted above, and is *increasing* for $g - nRx > G_{n-1}$, since by (7) and the inductive hypothesis the second summand in (10) is constant in this range. Thus the maximum of $P(x)$ required in (9) is obtained by setting $g - nRx = G_{n-1}$, and (9) becomes

$$M_n(g) = \frac{g - G_{n-1}}{nR} + M(G_{n-1}; g_2, \ldots, g_n) = \frac{g - G_{n-1}}{nR}$$

$$+ R^{-1} \left\{ \frac{G_{n-1}}{n-1} + \frac{G_1}{1 \cdot 2} + \ldots + \frac{G_{n-2}}{(n-2)(n-1)} \right\}$$

$$= R^{-1} \left\{ \frac{g}{n} + \frac{G_1}{1 \cdot 2} + \ldots \frac{G_{n-1}}{(n-1)n} \right\},$$

so that (6) is proved for $G_{n-1} < g \leq G_n$. Finally, (7) follows from (6) and the obvious fact that

$$M_n(g) = M_n(G_n) \qquad \text{if } g > G_n.$$

3. Equal Fuel Capacities

In this section it will be assumed that

$$g_i = G \qquad \text{for } i = 1, 2, \ldots, n \quad (12)$$

and that the aircraft are so numbered that

$$r_1 \geq r_2 \ldots \geq r_{n-1} \geq r_n > 0. \quad (13)$$

It is convenient to define

$$R_k = 1 / \sum_{j=1}^{k} r_{n+1-j} \qquad \text{for } k = 1, 2, \ldots, n. \quad (14)$$

The maximum range is denoted $M_n(g)$, or more fully by

$$M(g; r_1, \ldots, r_n) = M_n(g). \quad (15)$$

If $g \leq nG$, then there is a unique integer k for which

$$kG < g \leq (k+1)G. \quad (16)$$

The maximum range is given by

$$M_n(g) = G(R_1 + R_2 + \ldots + R_k) + R_{k+1}(g - kG)$$

$$\text{if } g \leq nG, \quad (17)$$

$$M_n(g) = G(R_1 + R_2 + \ldots + R_n) \qquad \text{if } g > nG. \quad (18)$$

In the optimal policy, if $g \leq nG$, the flight is begun with aircraft $n, n-1, \ldots, n-k$. If $g > nG$, then $g - nG$ gallons of fuel must be discarded and the flight is begun with all n aircraft. In any case, if the flight is begun with aircraft $n, n-1, \ldots, n-t$, then the last-listed of these aircraft is abandoned when only tG gallons of fuel are left, the next-to-last is abandoned when only $(t-1)G$ gallons remain, etc.

The proof of this result is just like that in the preceding section, and is therefore omitted. By way of orientation, it may be noted that the analog of (9) is

$$M_n(g) = \max_{x \leq R_n g} \{x + M(g - R_n^{-1}x; r_2, \ldots, r_n)\}.$$

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